2. LOGIC

§2.1. Propositional Logic

All of mathematics can be built on Set Theory. This is not the only possible foundation. The more recent study of Category Theory provides another approach. But each of these foundations rests on an even more fundamental ground – that of logic.

Indeed logic supports all of our rational thought. The surprising thing is that despite some snide remarks about 'feminine logic' there are no real cultural differences in logic. It seems that our logic is somehow hard-wired into our brains. Logicians have experimented with alternative logics but they still use the familiar naïve logic to analyse these exotic logics.

Like most of us, mathematicians take logic pretty much for granted, though they *do* make logic work much harder than most people.

The basic object in logic is a **statement** (or **proposition**). It's a sentence, possibly involving symbols, that can be validly assigned exactly one of the **truth values** T (TRUE) or F (FALSE). Sentences such as those that ask questions or give commands are obviously excluded. But we must also exclude self-referential sentences, even though they appears to be statements.

Take the sentence:

THIS STATEMENT IS FALSE

If it’s true then it’s false and if it’s false then it’s true!

Sometimes individual sentences may be valid statements, involving no self-referentiality, but taken together they lose that status.

Consider the following pair of sentences:

(1) THE SECOND STATEMENT IS TRUE
(2) THE FIRST STATEMENT IS FALSE

If (1) is true then (2) is true and so (1) is false, a contradiction. But if (1) is false then (2) is false and so (1) is true, again a contradiction. Here there is indirect self-referentiality.

But avoiding self-referentially is not enough. Consider the following infinite list of sentences:

(1) At least one of the following is FALSE.
(2) At least one of the following is FALSE.
(3) At least one of the following is FALSE.
(4) At least one of the following is FALSE.
(5) At least one of the following is FALSE.
………………………………………………...

At first sight it might appear that each sentence in the list is saying the same thing, but of course ‘the following’ refers to a decreasing set of sentences as we go down the list.

Clearly there’s no self-referentiality here, direct or indirect, in this list. But suppose that statement (n) is false. Then all of those that follow must be true, including statement ($n + 1$). But if statement ($n + 1$) is true then statement (m) is false for some $m > n + 1$, a contradiction. So each statement in the list must be true, also a contradiction.

It’s not an easy thing to define precisely what properties a sentence, or a collection of sentences, has to satisfy in order for them to be allowed to be considered as statements, so we just give up and hope for the best, confident that our mathematical intuition will guide us.

We postulate the existence of a collection of primitive statements and focus on compound statements that can be built up from them. The tools for doing this are called **logical operators**.

If p is a statement then **–p** denotes the statement ‘not p’.
If p and q are statements then:

p ∧ q denotes the statement that ‘both p and q’,
p ∨ q denotes the statement ‘p or q’,
p → q denotes ‘p implies q’,
p ↔ q means ‘p implies q and q implies p’.

§2.2. Quantifier Logic

A **predicate** is a statement that involves variables. Predicates become propositions when particular objects (such as integers) are substituted for the variables. The resulting propositions have truth values that depend on those elements.

Substituting for one variable in a predicate reduces the number of variables by 1. Substituting for all the variables produces a statement. Now there's another way of reducing the number of variables in a predicate and that is to quantify one or more variables.

Quantifiers will be familiar to you, even if you've never heard the word. In fact mathematics couldn't exist without them. There's the **universal quantifier**, denoted by ∀, and the **existential quantifier** ∃.

Suppose P*x* is a predicate involving one variable. The statement **∀*x*[P*x*]** means 'for all *x*, P*x*', that is, 'P*x* is TRUE for all *x*'.

The statement **∃*x*[P*x*]** means 'for some *x*, P*x* is TRUE'. This means that there's at least one *x* for which P*x* is TRUE. Sometimes it's read as "there exists *x* such that P*x*".

Behind a quantifier there's a non-empty **universe of quantification**. This is a class of objects from which the *x* comes. We refrain from calling it a *set*. What exactly is the difference between a 'set' and a 'class'? Many years ago Bertrand Russell drew attention to a certain paradox, which is now known as the **Russell Paradox**.

If one is allowed to talk about the set of all sets then we should be able to consider the subset consisting of all sets that are not elements of themselves.

Now the set of all integers is not an integer and the set of all functions of a real variable is not itself a function of a real variable. But the set of all sets, if we permitted such a thing, would be itself a set and so would be an element of itself.

So let $S = \{x \mid x \notin x\}$, with the universe of quantification to be the set of all sets. Now is $S \in S$? If the answer is "yes" then S has to satisfy the defining property for S, namely $S \notin S$. But this is a contradiction. If the answer is "no" then has to satisfy the negation of the defining property of S, that is $S \in S$, again a contradiction.

You may need to read this carefully several times. This is a complete contradiction. Notice that there is a hint of self-referentiality in the notion of the set of all sets.

So, not every adjective has a corresponding noun. Not every valid predicate can lead to a set. We can't contemplate such a thing as the set of all sets. Instead we call it a **class**.

Now this might seem to be just a clever sleight of hand, substituting a different word. But the important difference is that classes are collections which are not elements of other classes, unless they happen to be sets.

The class of all integers is also a set and so this set of integers can be an element of a larger set. But the class of all sets is what is known as a **proper class**. It's a class, but not a set.

So we can't have the universe of quantification be the class of all sets. So S = $\{x \mid x \notin x\}$ denotes the class of all sets that are not elements of themselves. All that the Russell Paradox now shows is that S is a proper class.

We'll state that a set is an undefined object, with membership, denoted by '$\in$', as an undefined relation between sets.

As we develop set theory, and on top of it the whole structure of mathematics, we must empty our minds of our intuitive concept of sets and set membership. Everything must be proved from the axioms, not by appealing to our innate concept of collections of things. Of course that doesn't stop us thinking of sets in the usual way. Intuition is a valuable tool in developing mathematics. However we must empty ourselves of this intuition when writing, or reading proofs. Every theorem in Set Theory must stand on the foundation of the set theory axioms. But, as we reflect on these theorems, we improve our intuitive concept of a set.

When we use the word 'class' and 'universe' we mean a set in the informal, intuitive sense while a 'set' is part of a collection of undefined objects in our Set Theory. There's nothing wrong with talking about the class of all sets, but if we attempt to make this class itself a set, within our system, we get a logical contradiction.

An ***n*-ary predicate** is one which involves *n* free variables. Each time a free variable is quantified the number of free variables is reduced by one. Special terms are: **unary** if *n* = 1, **binary** if *n* = 2 and **ternary** if *n* = 3. And, of course, if *n* = 0 it is a **statement**.

The following properties of quantifiers are intuitively obvious though, if we wanted to proceed more formally with our logic, we'd have to take some of them as axioms, and prove the others.

QUANTIFICATION RULES

(1) $-\forall x[Px] \leftrightarrow \exists x[-Px]$ If something isn't always TRUE it's sometimes FALSE.

(2) $-\exists x[Px] \leftrightarrow \forall x[-Px]$ If it's not the case that something can be TRUE it must always be FALSE.

(3) $\forall x Px \rightarrow \exists x Px$. If something is always TRUE then it is sometimes TRUE.

(4) $\forall x \forall y Pxy \leftrightarrow \forall y \forall x Pxy$.

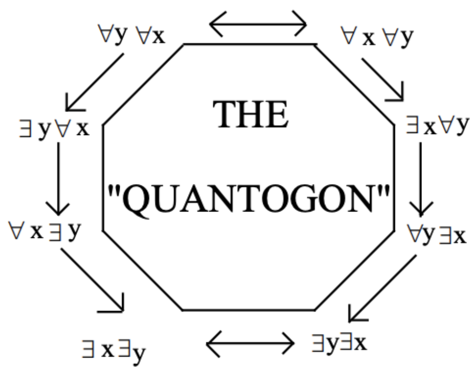
(5) $\exists x \exists y Pxy \leftrightarrow \exists y \exists x Pxy$.

(6) $\exists x \forall y Pxy \rightarrow \forall y \exists x Pxy$. If there's an *x* that makes P*xy* always TRUE then for all *x* there's a *y* that makes P*xy* TRUE.

You might have difficulty in seeing that (6) is intuitively obvious, so this example may help. If P*xy* was '*x* loves *y*' then $\forall y \exists x Pxy$ says that everyone has someone who loves them. But $\exists x \forall y Pxy$ is a stronger statement which has a theological flavour: there is someone who loves everybody. If you're an atheist you may not believe this stronger statement but you might believe the weaker one.

The explanations provided make these six rules seem plausible. However we'd need to accept them as axioms if we were doing our logic formally, or at least adopt (1), (3) (4) and (6) as axioms and prove (2) and (5) as consequences.

With a predicate involving two variables there are eight ways they can be quantified: each of *x*, *y* can be quantified by $\forall$ or $\exists$ and they can be quantified in either order. The logical relationships between them are displayed by this diagram.



§2.3. Negation Rules

To prove a theorem by contradiction requires the negation of the theorem to be assumed in order to reach a contradiction. If the statement has a complicated logical structure it may be necessary to rewrite this negation more simply. One can do this by using the following rules describing the way negation interacts with the other truth operators and the two quantifiers.

PROPOSITION	**NEGATION**
$-p$	p
$p \wedge q$	$-p \vee -q$
$p \vee q$	$-p \wedge -q$
$p \rightarrow q$	$p \wedge -q$
$p \leftrightarrow q$	$p \leftrightarrow -q$
$\forall x Px$	$\exists x -Px$
$\exists x Px$	$\forall x -Px$

§2.4. Relations and Functions

A unary predicate is a **property**, such as '*x* is even' or '*x* is female'.

A **relation** is a binary predicate, such as '$x < y$' or '*x* knows *y*'. We could write these symbolically as *x*L*y* for '$x < y$' and '*x*K*y*' for '*x* knows *y*'.

You may recall that the first time you met the word 'function' in mathematics it was a formula. Later you saw a more sophisticated definition: a function is a rule that associates with every *x* a unique value of *y*. There was no mention of sets. Later you learnt a more sophisticated definition that referred to domains and codomains.

A function $f : X \to Y$ is a pair of sets X and Y, called the **domain** and **codomain** respectively, together with a rule that associates with every $x \in X$ a unique $y \in Y$.

But we haven't yet created any sets and already we need to talk about functions in order to use them in one of our axioms. So we have to revert to a definition that doesn't mention sets. A **function** is a relation $x\mathrm{F}y$ such that $\forall x \forall y \forall z[x\mathrm{F}y \wedge x\mathrm{F}z \to y = z]$. The unique *y* such that $x\mathrm{F}y$ is denoted by $\mathrm{F}(x)$.

In the context of our set theory we limit relations, and hence functions, to those that can be expressed entirely in terms of the basic membership relation $x \in y$. Of course relations and functions that can be expressed in terms of relations that themselves can be expressed in terms of set membership will also be permitted.

For example we will define equality of sets in terms of them having precisely the same elements. That is:

$$\mathrm{S} = \mathrm{T} \text{ means } \forall x[x \in \mathrm{S} \leftrightarrow x \in \mathrm{T}]$$

So statements of the form $x = y$ are permitted in the definition of a set relation or function.